Special Relativity

Work Prior to Special Relativity

Consider two inertial reference frames, S and S', and suppose that S' moves with speed v with respect to S along the positive x axis. Assume that the axes of the two frames are parallel correspondingly. Suppose an event P happens and is measured by two observers in the two frames. **Galilean coordinate transformation** is classical, and the equations are

$$x' = x - vt \qquad y' = y \qquad z' = z \qquad t' = t$$

From these equations, we see that length and time are the same as measured in the two frames. In classical mechanics, we also assume that mass is independent of relative motion.

Given the above equations, we can differentiate both sides with respect to t. Noticing that $t = t'$, we get the equations about velocities,

$$u'_x = u_x - v \qquad u'_y = u_y \qquad u'_z = u_z$$

and similarly for accelerations,

$$a'_x = a_x \qquad a'_y = a_y \qquad a'_z = a_z$$

Since m is also assumed to be constant, we see that $\mathbf{F} = m\mathbf{a}$ is the same in all inertial reference frames. Therefore, we say that the laws of mechanics (Newton's laws, conservation laws, and so on) are the same for all inertial frames.

From Maxwell's equations, the speed of light is a constant, equal to $1/\sqrt{\mu_0\epsilon_0}$. But if electromagnetism accepts Galilean relativity, then the speed of light should not be a constant. There are several possibilities:

1. In electromagnetism there is a preferred reference frame, in which the speed of light is c.

2. Maxwell's electrodynamics theory is wrong.

3. Newton's mechanics is wrong, and so is Galilean transformation.

Regarding possibility 1, people believe that light propagates through a medium 'ether', or in other words, light cannot propagate in empty space. And this 'ether' corresponds to an absolute 'ether' frame in which the speed of light is c. However, the Michelson-Morley experiment showed that there is no such 'ether' frame, and the speed of light is the same for all directions in every inertial reference frame. So possibility 1 is actually impossible.

Hypotheses

1. The laws of physics are the same for all inertial reference frames.
2. The speed of light in vacant space is the same for all reference frames.

Note that the first hypothesis deals with all laws of physics, not only just laws of mechanics. In addition, in special relativity, we only consider inertial reference frames, but the objects of interest may be accelerating in these inertial frames. Also notice that an ideal rigid body does not exist in relativity, otherwise information can propagate at an infinite speed.

Concept of Simultaneity

If two events A, B are simultaneous, then two synchronized clocks, one at A and one at B, have the same reading when the two events happen. Note that how to synchronize the two clocks is a different problem.

Based on the two hypotheses, simultaneity is a relative concept. Suppose two events happen simultaneously as observed in one inertial frame S. Due to the finite speed of light, the two events do not happen simultaneously as observed in another inertial frame S' that is moving with a speed v relative to S, or in other words.

Notice that in the experiment there are 4 clocks, two for each frame. The simultaneity of the two events in frame S' depends only on the readings of the two clocks in S'.

Lorentz Transformation

Suppose that two inertial reference frames S, S' move with relative speed v along x axis, and the two axes x, x' overlap. Assume that $t = t' = 0$ when the two origins coincide. **Lorentz transformation** is given by

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \qquad y' = y \qquad z' = z \qquad t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}$$

An important quantity is the spacetime inverval, which is given by

$$s^2 = c^2t^2 - x^2 - y^2 - z^2$$

For an event, this quantity is invariant as measured in different inertial frames. Let τ be the proper time, then we have

$$s^2 = s'^2 = c^2\tau^2$$

Some consequences of the Lorentz transformation include length contraction and time dilation. Let $\beta = v/c$. Suppose in S' frame a rod is at rest and has endpoints x'_2 and x'_1. We measure the rod's length in S frame, with $t_2 = t_1$. Lorentz transformation gives

$$x_2 - x_1 = (x'_2 - x'_1)\sqrt{1 - \beta^2}$$

So in the S frame, the measured length of the rod becomes shorter.

Similarly, for time, we have

$$t_2 - t_1 = \frac{t'_2 - t'_1}{\sqrt{1-\beta^2}}$$

For the object of interest, the **proper frame** is the frame in which the object is at rest. Similar concepts include proper length and proper time interval. Notice that a proper time interval is measured by a single clock at a fixed position in the proper frame, while a nonproper time interval is measured by two synchronized clocks at two different positions in a different inertial frame.

Observer

An observer is really an infinite set of recording clocks distributed throughout space, at rest and synchronized with respect to one another.

Note that an observer does not mean measuring at only one or two points in the frame. Besides, an observer does not use eyes to 'see' things like length contraction, and an observer should not be confused with a viewer (a real person).

Addition of Velocities

Suppose S' frame is moving with velocity v relative to S frame, along the common x-x' axis. The Lorentz transformation gives

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2} \qquad u_y = \frac{u'_y\sqrt{1 - v^2/c^2}}{1 + u'_x v/c^2} \qquad u_z = \frac{u'_z\sqrt{1 - v^2/c^2}}{1 + u'_x v/c^2}$$

where u' is the velocity measured in S' frame, v the relative velocity between S and S', and u the velocity measured in S frame. Notice that since there is no length contraction in the perpendicular directions, the u'_x dependence of u_y and u_z is due to time dilation.

Doppler Effect

Again consider two frames S and S' with common x-x' axis. Suppose a light source is at rest in the S' frame, emitting plane waves of speed c in the $x'y'$ plane, and the wave direction is an angle θ' from the x' axis. In the S frame, we get

$$\tan\theta = \frac{\sin\theta'\sqrt{1-\beta^2}}{\cos\theta' + \beta}$$

The relativistic Doppler effect can also be derived from the above setup. The equation is

$$\nu = \frac{\nu'(1 + \beta\cos\theta')}{\sqrt{1-\beta^2}} = \frac{\nu'\sqrt{1-\beta^2}}{1 - \beta\cos\theta}$$

where ν and ν' are frequencies. The longitudinal Doppler effect corresponds to $\theta = 0$ and $\theta = 180°$. The transverse version corresponds to $\theta = 90°$ and $\theta = 270°$ (both yield $\cos\theta = 0$).

Relativistic Dynamics

Momentum

We want to see if we can preserve the form of the momentum of classical mechanics, that is, $\mathbf{p} = m\mathbf{v}$. After simple derivations, we find that to preserve the familiar form, the following modification is required:

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \tag{1}$$

where m_0 is the proper mass and m is the relativistic mass. Here v is the speed of the object measured in some inertial frame, and has nothing to do with relative motion between frames. With the modification of mass, the conservation of momentum in relativity reads

$$m_1\mathbf{v}_1 = m_2\mathbf{v}_2$$

Force

Accordingly, the force in relativity is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}\left(\frac{m_0\mathbf{v}}{\sqrt{1 - v^2/c^2}}\right)$$

Notice that the new force law is not equivalent to writing $\mathbf{F} = m\mathbf{a}$.

Energy

Suppose for simplicity, a particle moves in one dimension. Then relativistically, kinetic energy is

$$K = \int_{v=0}^{v=v} F dx = \int d(mv)\frac{dx}{dt} = \int (vdm + mdv)\, v = \int \left(v^2 dm + mvdv\right)$$

We rewrite eq.1 and differentiate both sides, and we get

$$v^2 dm + mvdv = c^2 dm$$

So we obtain

$$K = c^2 \int dm = mc^2 - m_0c^2 = m_0c^2\left(\frac{1}{\sqrt{1 - \beta^2}} - 1\right) \tag{2}$$

$E = mc^2$ is referred to as the total energy, and m_0c^2 is the rest energy. We have

$$E = m_0c^2 + K$$

We can also obtain relations between E and p from eq.1 and eq.2. The equation is

$$E^2 = (pc)^2 + (m_0c^2)^2$$

Acceleration

The general expression for acceleration is given by

$$\mathbf{a} = \frac{\mathbf{F}}{m} - \frac{\mathbf{v}}{mc^2}(\mathbf{F} \cdot \mathbf{v})$$

Equivalence of Mass and Energy

In relativity, the conservation of energy is equivalent to the conservation of (relativistic) mass. The transformation between them is given by the famous equation

$$E = mc^2$$

A photon is a 'particle' with zero rest mass and has speed c. This is consistent with the theory, and a speed of c in one inertial frame indicates that the speed is c in all inertial frames. With $m_0 = 0$, the energy of a photon is given by

$$E = pc$$

Transformation of Dynamics Quantities between Frames

Consider two inertial frames S and S' with a common x-x' axis, and relative speed v along the x axis. Then the transformations are

$$p'_x = \frac{p_x - Ev/c^2}{\sqrt{1 - v^2/c^2}} \qquad p'_y = p_y \qquad p'_z = p_z \qquad E' = \frac{E - vp_x}{\sqrt{1 - v^2/c^2}}$$

Comparing with the Lorentz transformation, we see that the above transformations are in the same form, if we identify the correspondence:

$$x, y, z \to p_x, p_y, p_z \qquad t \to E/c^2 = m$$

And an invariant quantity also exists and has the same form as the spacetime interval s.
The transformation of mass can be derived from the above equations, which is

$$m' = \frac{m(1 - u'_x v/c^2)}{\sqrt{1 - v^2/c^2}}$$

Let $\gamma = 1/\sqrt{1 - v^2/c^2}$, then the transformations of force are

$$F'_x = \frac{F_x - (v/c^2)\mathbf{u} \cdot \mathbf{F}}{1 - u_x v/c^2} \qquad F'_y = \frac{F_y}{\gamma(1 - u_x v/c^2)} \qquad F'_z = \frac{F_z}{\gamma(1 - u_x v/c^2)}$$

Relativistic Electromagnetism

Consider two inertial frames S and S', and S' move along the common x-x' axis with relative speed v. Then consider a cube of side length l_0 that is at rest in S' frame. Assume that there are N electron charges in the cube. By considering the charge density and current density in the two frames, we conclude find the following transformations:

$$\mathbf{j} = \frac{\rho_0 \mathbf{v}}{\sqrt{1 - v^2/c^2}} \qquad \rho = \frac{\rho_0}{\sqrt{1 - v^2/c^2}}$$

Notice the analogy between the three 4-element vectors: $(\mathbf{r}, t)$, $(\mathbf{p}, m)$, and $(\mathbf{j}, \rho)$. So the transformation of j and ρ has the familiar form.

Transformation of Electric Field

Again consider two inertial frames S and S' with relative speed v along the common x-x' axis. Suppose a charged particle is at rest in the S' frame, in which there are fields $\mathbf{E}'$ and $\mathbf{B}'$. Using the force transformations, we get

$$E'_x = E_x \qquad E'_y = \gamma(E_y - vB_z) \qquad E'_z = \gamma(E_z + vB_y)$$

Or more generally, we can write

$$E'_{\|} = E_{\|} \qquad E'_{\perp} = \gamma \left[E_{\perp} + (\mathbf{v} \times \mathbf{B})_{\perp}\right]$$

where $\|$ and $\perp$ mean the components parallel and perpendicular to the velocity $\mathbf{v}$.

Transformation of Magnetic Field

Using the same setup, except that now the charge is moving along the y' or z' axis in the S' frame, we can get the following general transformation:

$$B'_{\|} = B_{\|} \qquad B'_{\perp} = \gamma \left[B_{\perp} - \frac{1}{c^2}(\mathbf{v} \times \mathbf{E})_{\perp}\right]$$